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2965. Proposed by C. N. MILLS, Tiffin, Ohio.

If a quadrilateral inscribed in a square has the diagonals a and b, and the area A, show that the area of the square is $\dfrac{a^2b^2 - 4A^2}{a^2 + b^2 - 4A}$.

SOLUTIONS

2824 [1920, 185]. Proposed by G. Y. SOSNOW, Newark, N. J.

If n_1, n_2, n_3, n_4 be the lengths of the four normals and t_1, t_2, t_3 the lengths of the three tangents drawn from any point to the semi-cubical parabola, $ay^2 = x^3$, then will $27n_1n_2n_3n_4 = at_1t_2t_3$ [From *Mathematical Tripos Examination*, Cambridge, England].

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let the parametric equations of the curve be

$$x = au^2 \quad \text{and} \quad y = au^3.$$

Then

$$\frac{dy}{dx} = \frac{3u}{2},$$

the equations of the normal and tangent will be

$$3au^4 + 2au^2 - 3yu - 2x = 0 \tag{1}$$

and

$$au^3 - 3xu + 2y = 0, \tag{2}$$

and the lengths of the normal and tangent from (x, y) to the curve will be

$$n = \left(\frac{x - au^2}{3u}\right)\sqrt{9u^2 + 4} \quad \text{and} \quad t = \left(\frac{x - au^2}{2}\right)\sqrt{9u^2 + 4}.$$

The polynomial whose roots are the squares of the roots of (1) is easily found[1] to be:

$$9a^2z^4 + 12a^2z^3 + 4a(a - 3x)z^2 - (8ax + 9y^2)z + 4x^2 = 9a^2\Pi(z - u_i^2).$$

Multiply the roots by a, according to the familiar rule, and then replace z by x. This gives

$$\Pi(x - au_i^2) = x(x^3 - ay^2).$$

Also multiplying the roots by 9 and replacing z by -4 we obtain

$$\Pi(4 + 9u_i^2) = \frac{4}{a^2}[729(x^2 + y^2) + 216ax + 16a^2].$$

Hence

$$n_1n_2n_3n_4 = -\frac{x^3 - ay^2}{27}[729(x^2 + y^2) + 216ax + 16a^2]^{1/2}.$$

In a similar manner from equation (2) we get,

$$\Pi(x - au_i^2) = 4(x^3 - ay^2),$$

$$\Pi(4 + 9u_i^2) = \frac{4}{a^2}[729(x^2 + y^2) + 216ax + 16a^2];$$

and hence,

$$t_1t_2t_3 = \frac{x^3 - ay^2}{a}[729(x^2 + y^2) + 216ax + 16a^2]^{1/2}.$$

Therefore

$$at_1t_2t_3 = 27n_1n_2n_3n_4$$

neglecting the sign.

2834 [1920, 273]. Proposed by OTTO DUNKEL, Washington University.

In any triangle ABC let M and N be, respectively, the points in which the median and the bisector of the angle at A meet the side BC, Q and P the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to

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[1] See, for example, Todhunter, *An Elementary Treatise on the Theory of Equations*, London, 1880, p. 36; or Salmon, *Lessons Introductory to the Modern Higher Algebra*, Dublin, 1885, p. 350.—EDITORS.

BA meets AN produced. Prove that the straight line QO is perpendicular to BC and the similar theorem for the external bisector of the angle at A.

This proposition shows the relation between two constructions for the center of curvature O of a conic for which B and C are foci and A is a point of the conic. (The figure also gives an easy proof of the law of tangents for triangles; compare *1920,* 53–54, 465; *1921,* 71, 170. See also *1920,* 226.)

I. Solution by William Hoover, Columbus, Ohio.

In the usual notation of trilinear coördinates, the equations of AM and AN are, respectively,

$$\beta \sin B - \gamma \sin C = 0, \quad (1), \qquad \beta - \gamma = 0. \quad (2).$$

Since the equation of the side BC is $\alpha = 0$, the equation of PN has the form $\alpha + k(\beta - \gamma) = 0$, where k is to be determined by the fact that PN is perpendicular to AN. The condition for perpendicularity (Salmon's *Conic Sections,* 6th ed., p. 59) gives $2k \cos A/2 - \sin (B - C)/2 = 0$. The equation of PN is then

$$2\,\alpha \cos\frac{A}{2} + (\beta - \gamma) \sin\frac{B-C}{2} = 0. \tag{3}$$

Since PO passes through the intersection of (3) and $\gamma = 0$, its equation must have the form $2\alpha \cos A/2 + \beta \sin (B - C)/2 + l\gamma = 0$. The condition that PO is perpendicular to $\gamma = 0$ gives the value of l, and the final form of the equation of PO is

$$2\alpha \cos\frac{A}{2} + \beta \sin\frac{B-C}{2} + \gamma\left[\cos A \sin\frac{B-C}{2} + 2\cos B\cos\frac{A}{2}\right] = 0. \tag{4}$$

Using the same reasoning again, the equation of QO may be written in two forms

$$\beta \sin B - \gamma \sin C + k\left[2\alpha\cos\frac{A}{2} + (\beta - \gamma)\sin\frac{B-C}{2}\right]$$

$$\equiv l(\beta - \gamma) + m\left[2\alpha\cos\frac{A}{2} + \beta\sin\frac{B-C}{2} + \gamma\left(\cos A\sin\frac{B-C}{2} + 2\cos B\cos\frac{A}{2}\right)\right] = 0.$$

Equating the coefficients of α we have $k = m$, and after cancelling certain terms of the identity, it appears at once that $l = \sin B$. It follows then that $k \cos A/2 \cos (B - C)/2 = \sin (B - C)/2$. Inserting the value of k, we have for the equation of QO

$$\begin{aligned} 2\alpha\cos\frac{A}{2}\sin\frac{B-C}{2} &+ \beta\left[\sin^2\frac{B-C}{2} + \sin B\cos\frac{A}{2}\cos\frac{B-C}{2}\right] \\ &- \gamma\left[\sin^2\frac{B-C}{2} + \sin C\cos\frac{A}{2}\cos\frac{B-C}{2}\right] = 0. \end{aligned} \tag{5}$$

The test for the perpendicularity of (5) and $\alpha = 0$ gives after some reduction

$$2\cos\frac{A}{2}\sin\frac{B-C}{2}\left[1 - \sin^2\frac{B-C}{2} - \cos^2\frac{B-C}{2}\right] = 0,$$

and hence the proposition is proved for the internal bisector.

For the external bisector the equations are as follows:

$AN' : \beta + \gamma = 0, \qquad P'N' : 2\alpha \sin A/2 + (\beta + \gamma) \cos (B - C)/2 = 0,$

$P'O' : 2\alpha \sin A/2 + \beta \cos (B - C)/2 + \gamma[\cos A \cos (B - C)/2 + 2 \cos B \sin A/2] = 0,$

$$\begin{aligned} Q'O' : 2\alpha \sin A/2 \cos (B - C)/2 &+ \beta[\cos^2 (B - C)/2 + \sin B \sin A/2 \sin (B - C)/2] \\ &+ \gamma[\cos^2 (B - C)/2 - \sin C \sin A/2 \sin (B - C)/2] = 0. \end{aligned}$$

The test for the perpendicularity of $\alpha = 0$ and $Q'O'$ gives

$$2\sin\frac{A}{2}\cos\frac{B-C}{2}\left[1 - \cos^2\frac{B-C}{2} - \sin^2\frac{B-C}{2}\right] = 0.$$

II. Solution by T. M. Blakslee, Ames, Iowa.

Let the coördinates of A be $(0, 0)$, of B, $(c, 0)$, of C, (h, k), and let the lengths of the sides of the triangle be a, b, c. As the slope of BC is $k/(h - c)$, the theorem is proved if we show that the slope of OQ is $(c - h)/k$.

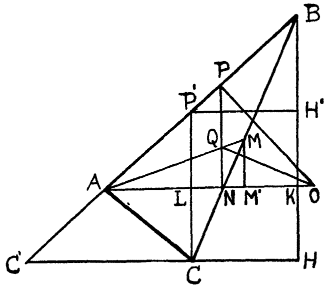
The coördinates of M are $[(c + h)/2, k/2]$, and the equation of AM is $y = kx/(c + h)$. The coördinates of N are $(\lambda b_1, \lambda k)$, where $b_1 = b + h$ and $\lambda = c/(b + c)$, and the equation of AN is $y = kx/b_1$. The equation of NP is $ky + b_1x = \lambda(k^2 + b_1^2)$, and hence the coördinates of P are $(\lambda s/b_1, 0)$ where $s = b_1^2 + k^2$. Since the x-coördinate for P and O are the same, we have for the coördinates of O, $(\lambda s/b_1, \lambda ks/b_1^2)$. The coördinates of Q are found from the equations of AM and NP to be $[\lambda s(c + h)/(k^2 + b_1c + b_1h), \lambda sk/(k^2 + b_1c + b_1h)]$. The slope of OQ is now found to be $[k^2 + b_1c - b_1(b_1 - h)]/(kb_1)$ and, using the relations $b_1 - h = b$ and $k^2 = b_1(b - h)$, this reduces to $(c - h)/k$. The proof for the external bisector may be carried through in the same manner.

III. Solution by the Proposer.

The truth of the theorem is evident if the angles B and C are equal. Suppose then that the angle B is the smaller of the two.

Prolong BA to C' so that $AC' = AC$. Then $C'C$ is parallel to AN. Let BH be the perpendicular from B upon $C'C$ produced to H. Draw CP' parallel to HB, cutting AN in L and AB in P'; also MM' parallel to HB meeting AN produced in M'. Draw $P'H'$ parallel to CH, meeting HB in H'. Then AN produced bisects HH' at K.

Since M is the middle point of BC, $M'M = (KB - KH)/2 = (KB - KH')/2 = H'B/2$ and $AM' = (AL + AK)/2 = C'H/2$. From similar triangles we have



$$\frac{NQ}{AN} = \frac{M'M}{AM'} = \frac{H'B}{C'H}, \quad \text{or} \quad \frac{NQ}{H'B} = \frac{AN}{C'H} = \frac{NP}{HB}.$$

Therefore

$$\frac{NQ}{NP} = \frac{H'B}{HB} = \frac{CH}{C'H}.$$

But

$$\frac{NP}{NO} = \frac{AN}{NP} = \frac{C'H}{HB}.$$

Therefore $NQ/NO = CH/HB$. Hence the right triangles ONQ, CHB are similar, and, since ON is perpendicular to HB, OQ must also be perpendicular to BC.

Also solved by P. J. da Cunha.

2851 [1920, 377]. Proposed by HILLEL PORITSKY, Cornell University.

Does there exist an analytic function satisfying the functional equation $f(z + 1) = e^{f(z)}$?

Solution by A. A. Bennett, University of Texas.

This is a typical problem of iteration and may be answered at once in the affirmative by reference to the usual methods in that theory.

It is necessary to find the fixed points of the iteration of e^x, that is, solutions of $x = e^x$. There are three such points; two proper finite points[1] ($.318 \pm i\,1.337$, approximately), and the point at infinity which latter satisfies the relation only for certain methods of approach. Let a denote one of the two finite fixed points above mentioned, then $a = e^a$, and the constant function a, is one solution of the problem. A one-parameter family of non-constant solutions including a as a member is readily found by the use of undetermined coefficients as follows:

Write $g(ax - a^2 + a) = e^{g(x)}$ where $g(ax - a^2 + a) = f(z + 1)$ and $g(x) = f(z)$; that is, $x = a^z + a$, and $g(x) = f[\log_a (x - a)]$. Now $g(x)$ may be expanded in the form,

$$g(x) = a + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots,$$

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[1] Compare this Monthly, *1921*, 141–142, footnote.